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# Integrals of motion of the Rabinovich system

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**Abstract.** In this paper, by using the method of characteristic curves for solving linear partial differential equations, we obtain the whole classification of the integrals of motion for the Rabinovich systems

$$\dot{x} = hy - v_1 x + yz \qquad \dot{y} = hx - v_2 y - xz \qquad \dot{z} = -v_3 z + xy.$$

## 1. Introduction and statement of the main results

We consider the Rabinovich system

$$\begin{aligned}\dot{x} &= hy - v_1 x + yz = P(x, y, z)\\ \dot{y} &= hx - v_2 y - xz = Q(x, y, z)\\ \dot{z} &= -v_3 z + xy = R(x, y, z)\end{aligned}$$

which is a three-wave interaction model, where $x$, $y$ and $z$ are real variables; $v_1$, $v_2$ and $v_3$ are the damping rates and $h$ is proportional to the driving amplitude of the feeder wave (see, for instance, [6] or [1]).

A real polynomial $f(x, y, z)$ is called a *Darboux polynomial* of the Rabinovich system if

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf \tag{1}$$

for some real polynomial $k(x, y, z)$, which is called the *cofactor* of $f$.

We say that a real function

$$H : \boldsymbol{R}^3 \times \boldsymbol{R} \longrightarrow \boldsymbol{R} \qquad (x, y, z, t) \longmapsto H(x, y, z, t)$$

is a *first integral* of the Rabinovich system if it is constant on all solution curves $(x(t), y(t), z(t))$ of the Rabinovich system, that is, $H(x(t), y(t), z(t), t) \equiv$ constant for all values of $t$ for which the solution $(x(t), y(t), z(t))$ is defined on $\boldsymbol{R}^3$. In particular, if the first integral $H$ is independent of the time and it is a polynomial, then it is called a *polynomial first integral*. If the first integral $H$ is of the form $f(x, y, z)\exp(kt)$, then it is called an *integral of motion*, where $f(x, y, z)$ is a polynomial, and $k$ is a real constant.

Using the Painlevé method in 1984 Bountis *et al* [1] found three integrals of motion as follows:

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- $I = (x^2 + y^2 - 4hz)e^{2vt}$ with $v_1 = v_2 = v > 0, v_3 = 2v, h \neq 0$;
- $I = (x^2 - y^2 - 2z^2)e^{2vt}$ with $v_1 = v_2 = v_3 = v > 0, h \neq 0$;
- $I = (x^2 + y^2)e^{2vt}$ with $v_1 = v_2 = v > 0, h = 0$.

In 1991, by making use of some algebraic methods, Giacomini *et al* [3] obtained the following four integrals of motion:

- $I = y^2 + (h - z)^2$ with $v_2 = v_3 = 0$;
- $I = x^2 - (z + h)^2$ with $v_1 = v_3 = 0$;
- $I = (y^2 + z^2)e^{2v_3 t}$ with $v_2 = v_3, h = 0$;
- $I = (x^2 - z^2)e^{2v_3 t}$ with $v_1 = v_3, h = 0$.

In this paper, by using the method of characteristic curves for solving linear partial differential equations, we characterize all integrals of motion. Our main result is the following.

**Theorem 1.** *The function $H(x, y, z, t)$ is an integral of motion for the Rabinovich system if and only if one of the following statements holds.*

*(a)* $v_1 = v_2 = v_3 = 0$*: the function*

$$H(x, y, z) = \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{m-s} (x^2 + y^2 - 4hz)^{m-s-i} (y^2 + z^2 - 2hz)^i$$

*is a polynomial first integral, where m is an arbitrary positive integer,* $\sum_{i=0}^{m} (a_i^m)^2 \neq 0$, *and* $a_i^{m-s}$ *is an arbitrary constant for* $s = 1, 2, \ldots, m$; $i = 0, 1, \ldots, m - s$.

*(b)* $v_1 = v_2 = v_3 \neq 0$ *and* $h = 0$*: the function*

$$H(x, y, z, t) = \sum_{i=0}^{m} a_i (x^2 + y^2)^{m-i} (y^2 + z^2)^i e^{2mv_1 t}$$

*is an integral of motion, where m is an arbitrary positive integer and* $\sum_{i=0}^{m} a_i^2 \neq 0$.

*(c)* $v_1 = v_2 = v_3 \neq 0$ *and* $h \neq 0$*: the function* $(x^2 - y^2 - 2z^2)^m e^{2mv_1 t}$ *is an integral of motion, where m is an arbitrary positive integer.*

*(d)* $v_1 = v_2 = 0$, $v_2 \neq v_3$ *and* $h = 0$*: the function* $H = \sum_{i=0}^{m} a_i (x^2 + y^2)^i$ *is a polynomial first integral, where m is an arbitrary positive integer and* $\sum_{i=0}^{m} a_i^2 \neq 0$.

*(e)* $v_1 = v_2 \neq 0$, $v_2 \neq v_3$ *and* $h = 0$*: the function* $H = (x^2 + y^2)^m e^{2mv_1 t}$ *is an integral of motion, where m is an arbitrary positive integer.*

*(f)* $v_1 = v_2 \neq 0$, $v_3 = 2v_1$ *and* $h \neq 0$*: the function* $H = (x^2 + y^2 - 4hz)^m e^{2mv_1 t}$ *is an integral of motion, where m is an arbitrary positive integer.*

*(g)* $v_1 \neq v_2$ *and* $v_2 = v_3 = 0$*: the function* $H = \sum_{i=1}^{m} a_i (y^2 + z^2 - 2hz)^i$ *is a polynomial first integral, where m is an arbitrary positive integer and* $\sum_{i=1}^{m} a_i^2 \neq 0$.

*(h)* $v_1 \neq v_2$, $v_2 = v_3 \neq 0$ *and* $h = 0$*: the function* $H = (y^2 + z^2)^m e^{2mv_2 t}$ *is an integral of motion, where m is an arbitrary positive integer.*

*(i)* $v_1 \neq v_2$, $v_2 \neq v_3$, $v_3 = v_1 \neq 0$ *and* $h = 0$*: the function* $(x^2 - z^2)^m e^{2mv_1 t}$ *is an integral of motion, where m is an arbitrary positive integer.*

*(j)* $v_1 \neq v_2$, $v_2 \neq v_3$, $v_3 = v_1 = 0$*: the function* $H = \sum_{i=1}^{m} a_i (x^2 - z^2 - 2hz)^i$ *is a polynomial first integral, where m is an arbitrary positive integer and* $\sum_{i=1}^{m} a_i^2 \neq 0$.

The following proposition shows the relationship between the Darboux polynomial and the integral of motion for the Rabinovich systems.

**Proposition 2.** *A Rabinovich system has a Darboux polynomial* $f(x, y, z)$ *with a constant cofactor k if and only if the function* $H(x, y, z, t) = f(x, y, z) \exp(-kt)$ *is a first integral.*

The proof of this proposition is easy, and follows in the same way as the proof of proposition 2 of [5], so we omit it. We note that from this proposition that, if we want to prove theorem 1, we only need to characterize all Darboux polynomials with the constant cofactor of the Rabinovich systems.

From theorem 1 and proposition 2, we easily obtain the following corollary.

**Corollary 3.** *(a) There are Rabinovich systems having irreducible polynomial first integrals of any even degree.*
*(b) The Rabinovich systems have no polynomial first integrals of odd degree.*

This paper is organized as follows. In section 2, we introduce the method of characteristic curves for solving linear partial differential equations: this is the main tool of this paper. In section 3, we prove theorem 1.

## 2. The method of characteristic curves

This section states the method of characteristic curves for solving linear partial differential equations (see, for instance, chapter 2 of [2]), which is a main tool of this paper.

Consider the following first-order linear partial differential equation:

$$a(x, y, z)A_x + b(x, y, z)A_y + c(x, y, z)A_z + d(x, y, z)A = f(x, y, z) \qquad (2)$$

where $A = A(x, y, z)$ and $a$, $b$, $c$, $d$ and $f$ are continuous differentiable.

A curve $(x(t), y(t), z(t))$ in the $xyz$-space is a *characteristic curve* for the partial differential equation (2) if, at each point $(x_0, y_0, z_0)$ on the curve, the vector $(a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0))$ is tangent to the curve. That is, the characteristic curve is a solution of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), z(t)) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), z(t)) \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = c(x(t), y(t), z(t)).$$

In practice, for convenience we treat $z$ as the independent variable instead of $t$, then the above system is reduced to the system (assuming $c(x, y, z) \neq 0$)

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{a(x, y, z)}{c(x, y, z)} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{b(x, y, z)}{c(x, y, z)}. \qquad (3)$$

This ordinary differential equation is known as the *characteristic equation* of (2).

Suppose that (3) has a solution in the implicit form $g(x, y, z) = c_1$, $h(x, y, z) = c_2$, where $c_1$ and $c_2$ are arbitrary constants. We consider the change of variables

$$u = g(x, y, z) \qquad v = h(x, y, z) \qquad w = z \qquad (4)$$

and we write its inverse transformation as $x = p(u, v, w)$, $y = q(u, v, w)$ and $z = r(u, v, w)$ (of course, sometimes the explicit inverse transformation cannot be obtained, or is not well defined). Then linear partial differential equation (2) becomes an ordinary differential equation in $w$ (for fixed $u$ and $v$)

$$\overline{c}(u, v, w)\overline{A}_w + \overline{d}(u, v, w)\overline{A} = \overline{f}(u, v, w) \qquad (5)$$

where $\overline{c}$, $\overline{d}$, $\overline{A}$ and $\overline{f}$ are $c$, $d$, $A$ and $f$, written in terms of $u$, $v$ and $w$.

If $\overline{A} = \overline{A}(u, v, w)$ is a solution of (5), then by transformation (4)

$$A(x, y, z) = \overline{A}(g(x, y, z), h(x, y, z), z)$$

is a solution of the linear partial differential equation (2). Moreover, the general solution of (5) is that of (2), written in terms of $x$, $y$ and $z$ by using (4).

## 3. The proof of theorem 1

The proof of the 'if' part follows from some straightforward calculations; the details are omitted. We now prove the 'only if' part.

From proposition 2, $H(x, y, z, t) = f(x, y, z)\mathrm{e}^{-kt}$ is an integral of motion for the Rabinovich system if and only if $f(x, y, z)$ is a Darboux polynomial with the constant cofactor $k$. We assume that

$$f(x, y, z) = \sum_{i=0}^{n} f_i(x, y, z)$$

is a Darboux polynomial of degree $n$ for the Rabinovich system with the constant cofactor $k(x, y, z) = c$, where $f_i$ is a homogeneous polynomial of degree $i$ for $i = 0, 1, \ldots, n$.

Substituting $f$ and $k = c$ into equation (1) and identifying the terms of the same degree, we obtain

$$yz\frac{\partial f_n}{\partial x} - xz\frac{\partial f_n}{\partial y} + xy\frac{\partial f_n}{\partial z} = 0 \tag{6}$$

$$yz\frac{\partial f_i}{\partial x} - xz\frac{\partial f_i}{\partial y} + xy\frac{\partial f_i}{\partial z} = (v_1 x - hy)\frac{\partial f_{i+1}}{\partial x} + (v_2 y - hx)\frac{\partial f_{i+1}}{\partial y} + v_3 z\frac{\partial f_{i+1}}{\partial z} + cf_{i+1} \tag{7}$$

for $i = n - 1, n - 2, \ldots, 1, 0$.

In what follows, in order to prove our theorem we will use the method of characteristic curves for solving linear partial differential equations. The characteristic equation associated with (6) is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{y}{x} \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = -\frac{y}{z}.$$

Its general solution is

$$x^2 + y^2 = c_1 \qquad y^2 + z^2 = c_2$$

where $c_1$ and $c_2$ are arbitrary constants.

We consider the change of variables

$$u = x^2 + y^2 \qquad v = y^2 + z^2 \qquad w = y. \tag{8}$$

Correspondingly, the inverse transformation is

$$x = \pm\sqrt{u - w^2} \qquad y = w \qquad z = \pm\sqrt{v - w^2}. \tag{9}$$

From equation (6) we obtain the ordinary differential equation

$$-\left(\pm\sqrt{u - w^2}\right)\left(\pm\sqrt{v - w^2}\right)\frac{\mathrm{d}\overline{f}_n}{\mathrm{d}w} = 0$$

where $\overline{f}_n(u, v, w) = f_n(x, y, z)$, and $u$ and $v$ are fixed. In the following, unless otherwise specified, we will always denote by $\overline{R}(u, v, w)$ the function $R(x, y, z)$, written in the variables $u$, $v$ and $w$ by using (9).

Solving this equation we obtain that

$$\overline{f}_n(u, v, w) = \overline{A}_n(u, v)$$

where $\overline{A}_n$ is an arbitrary function in $u$ and $v$. In order that $f_n(x, y, z) = \overline{f}_n(u, v, w) = \overline{A}(x^2 + y^2, y^2 + z^2)$ is a homogeneous polynomial of degree $n$ in $x$, $y$ and $z$, the integer $n$ must be even. Without loss of generality, we can assume that $n = 2m$, and that the general solution of (6) is

$$f_{2m} = \sum_{i=0}^{m} a_i^m (x^2 + y^2)^{m-i}(y^2 + z^2)^i$$

where $a_i^m$ is a real constant for $i = 0, 1, \ldots, m$.

Introducing $f_{2m}$ into equation (7) and performing some calculations, we have

$$
\begin{aligned}
yz\frac{\partial f_{2m-1}}{\partial x} - xz\frac{\partial f_{2m-1}}{\partial y} + xy\frac{\partial f_{2m-1}}{\partial z}
&= \sum_{i=0}^{m}[2(m-i)v_1 + 2iv_3 + c]a_i^m(x^2+y^2)^{m-i}(y^2+z^2)^i \\
&\quad + \sum_{i=0}^{m-1} 2[(m-i)(v_2-v_1)a_i^m + (i+1)(v_2-v_3)a_{i+1}^m] \\
&\quad \times (x^2+y^2)^{m-1-i}(y^2+z^2)^i y^2 \\
&\quad - \sum_{i=0}^{m} a_i^m \sum_{j=0}^{1}(4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j} \\
&\quad \times (x^2+z^2)^{m-1-i+j}(y^2+z^2)^{i-j}xy.
\end{aligned}
$$

Using the transformations (8) and (9), from this last equation we obtain the following ordinary differential equation:

$$
\begin{aligned}
\frac{\mathrm{d}\overline{f}_{2m-1}}{\mathrm{d}w} &= -\sum_{i=0}^{m}[2(m-i)v_1 + 2iv_3 + c]a_i^m u^{m-i}v^i \frac{1}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} \\
&\quad - \sum_{i=0}^{m-1} 2[(m-i)(v_2-v_1)a_i^m + (i+1)(v_2-v_3)a_{i+1}^m]u^{m-i-1}v^i \\
&\quad \times \frac{w^2}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} \\
&\quad + \sum_{i=0}^{m} a_i^m \sum_{j=0}^{1}(4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j} u^{m-1-i+j}v^{i-j}\frac{w}{\pm\sqrt{v-w^2}}.
\end{aligned}
$$

Integrating this equation with respect to $w$ we obtain

$$
\begin{aligned}
\overline{f}_{2m-1} &= -\sum_{i=0}^{m}[2(m-i)v_1 + 2iv_3 + c]a_i^m u^{m-i}v^i \int \frac{\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} \\
&\quad - \sum_{i=0}^{m-1} 2[(m-i)(v_2-v_1)a_i^m + (i+1)(v_2-v_3)a_{i+1}^m] \\
&\quad \times u^{m-i-1}v^i \int \frac{w^2\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} \\
&\quad - \sum_{i=0}^{m} a_i^m \sum_{j=0}^{1}(4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j} u^{m-1-i+j}v^{i-j}\left(\pm\sqrt{v-w^2}\right) \\
&\quad + \overline{f}^*_{2m-1}(u, v)
\end{aligned}
$$

where $\overline{f}^*_{2m-1}$ is an arbitrary function in $u$ and $v$.

An easy computation gives

$$
\int \frac{w^2\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}} = -\int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}\,\mathrm{d}w + u\int \frac{\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}}.
$$

Since

$$\int \frac{\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}} \qquad \text{and} \qquad \int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}\,\mathrm{d}w$$

are elliptic integrals of the first and second kind respectively (see, for instance, [4]), in order that $f_{2m-1}$ is a homogeneous polynomial of degree $2m-1$, we must have $\overline{f}^*_{2m-1}(x^2+z^2, y^2+z^2) \equiv 0$ and

$$\begin{array}{ll} [2(m-i)v_1 + 2iv_3 + c]a_i^m = 0 & i = 0, 1, \ldots, m \\ (m-i)(v_2 - v_1)a_i^m + (i+1)(v_2 - v_3)a_{i+1}^m = 0 & i = 0, 1, \ldots, m-1. \end{array} \tag{10}$$

Therefore,

$$\begin{aligned} f_{2m-1} &= -\sum_{i=0}^{m} a_i^m \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-1-i+j}(y^2+z^2)^{i-j} z \\ &= -\sum_{i=0}^{m-1} [4h(m-i)a_i^m + 2h(i+1)a_{i+1}^m](x^2+y^2)^{m-1-i}(y^2+z^2)^i z. \end{aligned} \tag{11}$$

From equations (10) we distinguish the following four cases:

(i) $v_1 = v_2 = v_3$, and then $c = -2mv_1$;
(ii) $v_1 = v_2$, $v_2 \neq v_3$, and then $a_1^m = a_2^m = \cdots = a_m^m = 0$, $a_0^m \neq 0$ and $c = -2mv_1$;
(iii) $v_1 \neq v_2$, $v_2 = v_3$, and then $a_0^m = a_1^m = \cdots = a_{m-1}^m = 0$, $a_m^m \neq 0$ and $c = -2mv_2$;
(iv) $v_1 \neq v_2$, $v_2 \neq v_3$, and then $v_1 = v_3$, $c = -2mv_1$ and $a_i^m \neq 0$ for $i = 0, 1, \ldots, m$.

*Case (i).* $v_1 = v_2 = v_3$ and $c = -2mv_1$. Introducing $f_{2m-1}$ into equation (7) with $i = 2m-2$ and performing some calculations, we obtain

$$\begin{aligned} & yz\frac{\partial f_{2m-2}}{\partial x} - xz\frac{\partial f_{2m-2}}{\partial y} + xy\frac{\partial f_{2m-2}}{\partial z} \\ &\quad = \sum_{i=0}^{m-1} 2hv_1[2(m-i)a_i^m + (i+1)a_{i+1}^m](x^2+y^2)^{m-1-i}(y^2+z^2)^i z \\ &\quad + \sum_{i=0}^{m} a_i^m 2 \sum_{j=0}^{2} (4h)^{2-j}\binom{m-i}{2-j}(2h)^j \binom{i}{j} \\ &\quad \times (x^2+y^2)^{m-2-i+j}(y^2+z^2)^{i-j} xyz. \end{aligned}$$

In the above computations, we used the following.

**Lemma 4.** *For any non-negative integers m, s and i satisfying $m > s+i$, the following equality holds:*

$$\begin{aligned} &\sum_{j=0}^{s} (4h)^{s-j}\binom{m-i}{s-j}(2h)^j\binom{i}{j} 4h(m-s-i+j)(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j} \\ &\quad + \sum_{j=0}^{s} (4h)^{s-j}\binom{m-i}{s-j}(2h)^j\binom{i}{j} 2h(i-j)(x^2+y^2)^{m-s-i+j}(y^2+z^2)^{i-1-j} \\ &\quad = (s+1)\sum_{j=0}^{s+1} (4h)^{s+1-j}\binom{m-i}{s+1-j}(2h)^j\binom{i}{j}(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}. \end{aligned}$$

**Proof.** By straightforward computations we have

$$
\begin{aligned}
&\sum_{j=0}^{s}(4h)^{s-j}\binom{m-i}{s-j}(2h)^j\binom{i}{j}4h(m-s-i+j)(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}\\
&\qquad+\sum_{j=0}^{s}(4h)^{s-j}\binom{m-i}{s-j}(2h)^j\binom{i}{j}2h(i-j)\\
&\qquad\times(x^2+y^2)^{m-s-i+j}(y^2+z^2)^{i-1-j}\\
&\quad=(4h)^{s+1}\binom{m-i}{s}(m-s-i)(x^2+y^2)^{m-1-s-i}(y^2+z^2)^{i}\\
&\qquad+\sum_{j=1}^{s}(4h)^{s+1-j}\binom{m-i}{s-j}(2h)^j\binom{i}{j}(m-s-i+j)\\
&\qquad\times(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}\\
&\qquad+\sum_{j=0}^{s-1}(4h)^{s-j}\binom{m-i}{s-j}(2h)^{j+1}\binom{i}{j}(i-j)\\
&\qquad\times(x^2+y^2)^{m-s-i+j}(y^2+z^2)^{i-1-j}\\
&\qquad+(2h)^{s+1}\binom{i}{s}(i-s)(x^2+y^2)^{m-i}(y^2+z^2)^{i-1-s}\\
&\quad=(s+1)(4h)^{s+1}\binom{m-i}{s+1}(x^2+y^2)^{m-1-s-i}(y^2+z^2)^{i}\\
&\qquad+\sum_{j=1}^{s}(s+1-j)(4h)^{s+1-j}\binom{m-i}{s+1-j}(2h)^j\binom{i}{j}\\
&\qquad\times(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}\\
&\qquad+\sum_{j=1}^{s}(4h)^{s+1-j}\binom{m-i}{s+1-j}j(2h)^j\binom{i}{j}\\
&\qquad\times(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}\\
&\qquad+(s+1)(2h)^{s+1}\binom{i}{s+1}(x^2+y^2)^{m-i}(y^2+z^2)^{i-1-s}\\
&\quad=(s+1)\sum_{j=0}^{s+1}(4h)^{s+1-j}\binom{m-i}{s+1-j}(2h)^j\binom{i}{j}\\
&\qquad\times(x^2+y^2)^{m-1-s-i+j}(y^2+z^2)^{i-j}.
\end{aligned}
$$

This proves the lemma. □

From the previous equation in $f_{2m-2}$ we obtain the following ordinary differential equation, taking into account the changes (8) and (9):

$$
\begin{aligned}
\frac{\overline{f}_{2m-2}}{\mathrm{d}w}=&-\sum_{i=0}^{m-1}2hv_1[2(m-i)a_i^m+(i+1)a_{i+1}^m]u^{m-1-i}v^i\frac{1}{\pm\sqrt{u-w^2}}\\
&-\sum_{i=0}^{m}a_i^m 2\sum_{j=0}^{2}(4h)^{2-j}\binom{m-i}{2-j}(2h)^j\binom{i}{j}u^{m-2-i+j}v^{i-j}w.
\end{aligned}
$$

Since

$$\int \frac{\mathrm{d}w}{\sqrt{u - w^2}} = \arcsin\left(\frac{w}{\sqrt{u}}\right) \tag{12}$$

in order that $f_{2m-2}(x, y, z) = \overline{f}_{2m-2}(u, v, w)$ is a homogeneous polynomial in $x$, $y$ and $z$, we must have

$$hv_1[2(m-i)a_i^m + (i+1)a_{i+1}^m] = 0 \qquad \text{for} \quad i = 0, 1, \ldots, m-1. \tag{13}$$

Therefore,

$$\begin{aligned} f_{2m-2} = &-\sum_{i=0}^{m} a_i^m \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} (x^2+z^2)^{m-2-i+j} (y^2+z^2)^{i-j} y^2 \\ &+ f_{2m-2}^*(x^2+y^2, y^2+z^2) \\ = &\sum_{i=0}^{m} a_i^m \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} (x^2+z^2)^{m-2-i+j} (y^2+z^2)^{i-j} z^2 \\ &- \sum_{i=0}^{m} a_i^m \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} (x^2+z^2)^{m-2-i+j} (y^2+z^2)^{i+1-j} \\ &+ f_{2m-2}^*(x^2+y^2, y^2+z^2) \end{aligned}$$

where $f_{2m-2}^*$ is an arbitrary function in $x^2 + y^2$ and $y^2 + z^2$. Without loss of generality, we select

$$\begin{aligned} f_{2m-2} = &\sum_{i=0}^{m} a_i^m \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} (x^2+z^2)^{m-2-i+j} (y^2+z^2)^{i-j} z^2 \\ &+ \sum_{i=0}^{m-1} a_i^{m-1} (x^2+y^2)^{m-1-i} (y^2+z^2)^i \end{aligned}$$

where $a_i^{m-1}$ is a real constant for $i = 0, 1, \ldots, m-1$. From condition (13) we distinguish the following three cases.

*Subcase 1.* $h = 0$. Then we have

$$f_{2m-1} \equiv 0 \qquad f_{2m-2} = \sum_{i=0}^{m-1} a_i^{m-1} (x^2+y^2)^{m-1-i} (y^2+z^2)^i.$$

Introducing $f_{2m-2}$ into equation (7) with $i = 2m-3$ and performing some computations, we obtain

$$yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z} = -2v_1 \sum_{i=0}^{m-1} a_i^{m-1} (x^2+y^2)^{m-1-i} (y^2+z^2)^i.$$

Using the transformations (8) and (9) and working in a similar way to solving $f_{2m-1}$, we obtain

$$\frac{\mathrm{d}\overline{f}_{2m-3}}{\mathrm{d}w} = 2v_1 \sum_{i=0}^{m-1} a_i^{m-1} u^{m-1-i} v^i \frac{1}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)}.$$

Similar to the proof of $f_{2m-1}$, in order that $f_{2m-3}$ is a homogeneous polynomial of degree $2m-3$ we must have

$$v_1 a_i^{m-1} = 0 \qquad \text{for} \quad i = 0, 1, \ldots, m-1 \tag{14}$$

and $f_{2m-3} \equiv 0$. By recursive calculations, we obtain that for $s = 2, 3, \ldots, m-1$

$$f_{2m-2s} = \sum_{i=0}^{m-s} a_i^{m-s}(x^2+y^2)^{m-s-i}(y^2+z^2)^i \qquad f_{2m-2s-1} \equiv 0$$

with conditions

$$v_1 a_i^{m-s} = 0 \qquad \text{for} \quad s = 2, 3, \ldots, m-1 \qquad i = 0, 1, \ldots, m-s. \tag{15}$$

If $v_1 = 0$, then $c = v_1 = v_2 = v_3 = 0$. By (14) and (15) we obtain that

$$f = \sum_{s=0}^{m-1}\sum_{i=0}^{m-s} a_i^{m-s}(x^2+y^2)^{m-s-i}(y^2+z^2)^i$$

is a polynomial first integral of degree $2m$, where $\sum_{i=0}^{m}[a_i^m]^2 \neq 0$ and $a_i^{m-s}$ is an arbitrary constant for $s = 1, 2, \ldots, m-1$ and $i = 0, 1, \ldots, m-s$. This proves statement (a) with $h = 0$ of theorem 1.

If $v_1 \neq 0$, then $a_i^{m-s} = 0$ for $s = 1, 2, \ldots, m-1$ and $i = 0, 1, \ldots, m-s$. Hence

$$f = \sum_{i=0}^{m} a_i^m (x^2+y^2)^{m-i}(y^2+z^2)^i$$

is a Darboux polynomial with the constant cofactor $k = -2mv_1$, where $\sum_{i=0}^{m}\left[a_i^m\right]^2 \neq 0$. This proves statement (b) of theorem 1.

*Subcase 2.* $h \neq 0$ and $v_1 = 0$. Then $v_1 = v_2 = v_3 = c = 0$. Substituting $f_{2m-2}$ into equation (7) with $i = 2m-3$ and performing some calculations which are similar to the proof of $f_{2m-1}$, we have

$$\begin{aligned} yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z} = &-\sum_{i=0}^{m} a_i^m 3\sum_{j=0}^{3}(4h)^{3-j}\binom{m-i}{3-j}(2h)^j\binom{i}{j} \\ &\times(x^2+y^2)^{m-3-i+j}(y^2+z^2)^{i-j}xyz^2 \\ &-\sum_{i=0}^{m-1} a_i^{m-1}\sum_{j=0}^{1}(4h)^{1-j}\binom{m-1-i}{1-j}(2h)^j\binom{i}{j} \\ &\times(x^2+y^2)^{m-2-i+j}(y^2+z^2)^{i-j}xy. \end{aligned}$$

Using the transformations (8) and (9), from this partial differential equation we obtain the following ordinary differential equation:

$$\begin{aligned} \frac{d\overline{f}_{2m-3}}{dw} = &\sum_{i=0}^{m} a_i^m 3\sum_{j=0}^{3}(4h)^{3-j}\binom{m-i}{3-j}(2h)^j\binom{i}{j}u^{m-3-i+j}v^{i-j}w\left(\pm\sqrt{v-w^2}\right) \\ &+\sum_{i=0}^{m-1} a_i^{m-1}\sum_{j=0}^{1}(4h)^{1-j}\binom{m-1-i}{1-j}(2h)^j\binom{i}{j}u^{m-2-i+j}v^{i-j}\frac{w}{\pm\sqrt{v-w^2}}. \end{aligned}$$

Integrating this equation with respect to $w$ and in a similar way to the proof of $f_{2m-1}$, we obtain

$$\begin{aligned} f_{2m-3} = &-\sum_{i=0}^{m} a_i^m \sum_{j=0}^{3}(4h)^{3-j}\binom{m-i}{3-j}(2h)^j\binom{i}{j}(x^2+y^2)^{m-3-i+j}(y^2+z^2)^{i-j}z^3 \\ &-\sum_{i=0}^{m-1} a_i^{m-1}\sum_{j=0}^{1}(4h)^{1-j}\binom{m-1-i}{1-j}(2h)^j\binom{i}{j} \\ &\times(x^2+y^2)^{m-2-i+j}(y^2+z^2)^{i-j}z. \end{aligned}$$

Introducing $f_{2m-3}$ into equation (7) with $i = 2m - 4$ and performing some calculations which are similar to the proof of $f_{2m-2}$, we have

$$yz\frac{\partial f_{2m-4}}{\partial x} - xz\frac{\partial f_{2m-4}}{\partial y} + xy\frac{\partial f_{2m-4}}{\partial z}$$

$$= \sum_{i=0}^{m} a_i^m 4 \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-4-i+j}(y^2+z^2)^{i-j} xyz^3$$
$$+ \sum_{i=0}^{m-1} a_i^{m-1} 2 \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-3-i+j}(y^2+z^2)^{i-j} xyz.$$

By using the changes (8) and (9) and working in a similar way to the proof of $f_{2m-2}$ and $f_{2m-3}$, we obtain that

$$\overline{f}_{2m-4} = \sum_{i=0}^{m} a_i^m \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j} u^{m-4-i+j} v^{i-j} (v-w^2)^2$$
$$- \sum_{i=0}^{m-1} a_i^{m-1} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j} u^{m-3-i+j} v^{i-j} w^2$$
$$+ \overline{A}_{2m-4}(u, v)$$

where $\overline{A}_{2m-4}$ is an arbitrary function in $u$ and $v$. Therefore, from the change (8) we have

$$f_{2m-4} = \sum_{i=0}^{m} a_i^m \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-4-i+j}(y^2+z^2)^{i-j} z^4$$
$$+ \sum_{i=0}^{m-1} a_i^{m-1} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-3-i+j}(y^2+z^2)^{i-j} z^2$$
$$- \sum_{i=0}^{m-1} a_i^{m-1} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-3-i+j}(y^2+z^2)^{i-j+1} + \overline{A}_{2m-4}(x^2+y^2, y^2+z^2).$$

In order that $f_{2m-4}$ is a homogeneous polynomial of degree $2m-4$, without loss of generality we can select

$$f_{2m-4} = \sum_{i=0}^{m} a_i^m \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-4-i+j}(y^2+z^2)^{i-j} z^4$$
$$+ \sum_{i=0}^{m-1} a_i^{m-1} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-3-i+j}$$
$$\times (y^2+z^2)^{i-j} z^2 + \sum_{i=0}^{m-2} a_i^{m-2} (x^2+y^2)^{m-2-i}(y^2+z^2)^i.$$

By recursive calculations, we can obtain that for $l = 2, 3, \ldots, m-1$

$$f_{2m-2l} = \sum_{i=0}^{m} a_i^m \sum_{j=0}^{2l} (4h)^{2l-j} \binom{m-i}{2l-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-2l-i+j}(y^2+z^2)^{i-j} z^{2l}$$

$$+\sum_{i=0}^{m-1} a_i^{m-1} \sum_{j=0}^{2l-2} (4h)^{2l-2-j} \binom{m-1-i}{2l-2-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-2l+1-i+j} (y^2+z^2)^{i-j} z^{2l-2}$$
$$+\sum_{i=0}^{m-2} a_i^{m-2} \sum_{j=0}^{2l-4} (4h)^{2l-4-j} \binom{m-2-i}{2l-4-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-2l+2-i+j} (y^2+z^2)^{i-j} z^{2l-4}$$
$$+\cdots+\sum_{i=0}^{m-l+1} a_i^{m-l+1} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-l+1-i}{2-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-l-1-i+j} (y^2+z^2)^{i-j} z^2 + \sum_{i=0}^{m-l} a_i^{m-l} (x^2+y^2)^{m-l-i} (y^2+z^2)^i$$
$$= \sum_{s=0}^{l} \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{2l-2s} (4h)^{2l-2s-j} \binom{m-s-i}{2l-2s-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-2l+s-i+j} (y^2+z^2)^{i-j} z^{2l-2s}.$$
$$f_{2m-2l-1} = -\sum_{s=0}^{l} \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{2l-2s+1} (4h)^{2l+1-2s-j} \binom{m-s-i}{2l+1-2s-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-2l-1+s-i+j} (y^2+z^2)^{i-j} z^{2l+1-2s}.$$

Combining these two expressions we have

$$f_{2m-2l-1} = (-1)^l \sum_{s=0}^{[l/2]} \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{l-2s} (4h)^{l-2s-j} \binom{m-s-i}{l-2s-j} (2h)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-l+s-i+j} (y^2+z^2)^{i-j} z^{l-2s}.$$

Hence, for the Rabinovich system the Darboux polynomial of degree $2m$ is

$$f = \sum_{l=0}^{2m-1} f_{2m-l} = \sum_{l=0}^{2m-1} \sum_{s=0}^{[l/2]} \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{l-2s} (-4hz)^{l-2s-j} \binom{m-s-i}{l-2s-j} (-2hz)^j \binom{i}{j}$$
$$\times (x^2+y^2)^{m-l+s-i+j} (y^2+z^2)^{i-j}.$$

For every given $s$ $(0 \leqslant s < m)$ the sum of the terms $a_i^{m-s}$ with the same superscript $m-s$ is

$$\sum_{i=0}^{m-s} a_i^{m-s} \sum_{l=2s}^{2m-1} \sum_{j=0}^{l-2s} \binom{m-s-i}{l-2s-j} (x^2+y^2)^{m-l+s-i+j} (-4hz)^{l-2s-j}$$
$$\times \binom{i}{j} (y^2+z^2)^{i-j} (-2hz)^j$$
$$= \sum_{i=0}^{m-s} a_i^{m-s} \sum_{t=0}^{2m-1-2s} \sum_{j=0}^{t} \binom{m-s-i}{t-j} (x^2+y^2)^{m-s-i-(t-j)} (-4hz)^{t-j}$$
$$\times \binom{i}{j} (y^2+z^2)^{i-j} (-2hz)^j$$
$$= \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{i} \sum_{t=j}^{2m-1-2s} \binom{m-s-i}{t-j} (x^2+y^2)^{m-s-i-(t-j)} (-4hz)^{t-j}$$

$$\times \binom{i}{j}(y^2+z^2)^{i-j}(-2hz)^j$$

$$= \sum_{i=0}^{m-s} a_i^{m-s} \sum_{j=0}^{i} \sum_{t=0}^{2m-1-2s-j} \binom{m-s-i}{t-j} (x^2+y^2)^{m-s-i-t}(-4hz)^t$$

$$\times \binom{i}{j}(y^2+z^2)^{i-j}(-2hz)^j$$

$$= \sum_{i=0}^{m-s} a_i^{m-s} \sum_{t=0}^{m-s-i} \binom{m-s-i}{t}$$

$$\times (x^2+y^2)^{m-s-i-t}(-4hz)^t \sum_{j=0}^{i} \binom{i}{j}(y^2+z^2)^{i-j}(-2hz)^j$$

$$= \sum_{i=0}^{m-s} a_i^{m-s}(x^2+y^2-4hz)^{m-s-i}(y^2+z^2-2hz)^i.$$

So, we obtain that

$$f = \sum_{s=0}^{m-1}\sum_{i=0}^{m-s} a_i^{m-s}(x^2+y^2-4hz)^{m-s-i}(y^2+z^2-2hz)^i$$

is a polynomial first integral of degree $2m$, where $\sum_{i=0}^{m} a_i^m \neq 0$. This proves the statement (a) of theorem 1.

*Subcase 3.* $h \neq 0$ and $v_1 \neq 0$. Then from condition (13) we have $2(m-i)a_i^m + (i+1)a_{i+1}^m = 0$ for $i = 0, 1, \ldots, m-1$, that is

$$a_i^m = (-2)^i \binom{m}{i} a_0^m.$$

Hence, we obtain that

$$f_{2m} = \sum_{i=0}^{m} a_i^m (x^2+y^2)^{m-i}(y^2+z^2)^i = \sum_{i=0}^{m} (-2)^i \binom{m}{i} a_0^m (x^2+y^2)^{m-i}(y^2+z^2)^i$$

$$= a_0^m[(x^2+y^2) - 2(y^2+z^2)]^2 = a_0^m(x^2-y^2-2z^2)^m$$

$f_{2m-1} \equiv 0.$

From equation (7) with $i = 2m-2$ and working in a similar way to solve $f_{2m}$, we can easily obtain that

$$f_{2m-2} = \sum_{i=0}^{m-1} a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^i.$$

Inserting $f_{2m-2}$ into equation (7) with $i = 2m-3$ and performing some computations, we have

$$yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z} = -\sum_{i=0}^{m-1} 2v_1 a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^i$$

$$= -\sum_{i=0}^{m-2}[4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^m](x^2+y^2)^{m-2-i}(y^2+z^2)^i xy.$$

Using the changes of the variables (8) and (9), from this equation we obtain the following ordinary differential equation:

$$\frac{\mathrm{d}\overline{f}_{2m-3}}{\mathrm{d}w} = \sum_{i=0}^{m-1} 2v_1 a_i^{m-1} u^{m-1-i} v^i \frac{1}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)}$$
$$+\sum_{i=0}^{m-2} [4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^m] u^{m-2-i} v^i \frac{w}{\pm\sqrt{v-w^2}}.$$

Integrating this equation with respect to $w$, we obtain

$$\overline{f}_{2m-3} = \sum_{i=0}^{m-1} 2v_1 a_i^{m-1} u^{m-1-i} v^i \int \frac{\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)}$$
$$-\sum_{i=0}^{m-2} [4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^m] u^{m-2-i} v^i z + \overline{A}_{2m-3}(u, v)$$

where $\overline{A}_{2m-3}$ is an arbitrary function in $u$ and $v$. So, in order that $f_{m-1}(x, y, z) = \overline{f}_{m-1}(u, v, w)$ is a homogeneous polynomial of degree $2m-3$, we must have $\overline{A}_{2m-3}(x^2 + y^2, y^2 + z^2) \equiv 0$ and $v_1 a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-1$. This means that $a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-1$. Moreover, we obtain that $f_{2m-3} \equiv 0$.

By recursive calculations and in a similar way to solving $f_{2m-2}$ and $f_{2m-3}$, we can obtain that $f_{2m-l} \equiv 0$ for $l = 4, 5, \ldots, 2m$. Therefore, the function

$$f = \sum_{i=0}^{2m} f_{2m} = a_0^m (x^2 - y^2 - 2z^2)^m$$

is a Darboux polynomial of degree $2m$ with the constant cofactor $-2mv_1$ for the Rabinovich system. This proves statement (c) of theorem 1.

*Case (ii).* $v_1 = v_2$ and $v_2 \neq v_3$. Then

$$f_{2m} = a_0^m (x^2 + y^2)^m \qquad f_{2m-1} = -4hma_0^m (x^2 + y^2)^{m-1} z.$$

Introducing $f_{2m-1}$ into (7) with $i = 2m-2$ and performing some computations, we obtain

$$yz\frac{\partial f_{2m-2}}{\partial x} - xz\frac{\partial f_{2m-2}}{\partial y} + xy\frac{\partial f_{2m-2}}{\partial z} = -4(v_3 - 2v_1)mha_0^m (x^2 + y^2)^{m-1} z$$
$$+2(4h)^2 \binom{m}{2} a_0^m (x^2 + y^2)^{m-2} xyz.$$

By using the changes (8) and (9) and working in a similar way to case (i), we obtain

$$\overline{f}_{2m-2} = 4(v_3 - 2v_1)mha_0^m u^{m-1} \arcsin\frac{w}{\sqrt{u}} - (4h)^2 \binom{m}{2} a_0^m u^{m-2} w^2 + \overline{A}_{2m-2}(u, v)$$

where $\overline{A}_{2m-2}$ is an arbitrary function in $u$ and $v$. In order that $f_{2m-2}(x, y, z)$ is a homogeneous polynomial of degree $2m-2$, we must have $h(v_3 - 2v_1) = 0$ and

$$f_{2m-2} = (4h)^2 \binom{m}{2} a_0^m (x^2 + y^2)^{m-2} z^2 + \sum_{i=0}^{m-1} a_i^{m-1} (x^2 + y^2)^{m-1-i} (y^2 + z^2)^i$$

where $a_i^{m-1}$ are real constants for $i = 0, 1, \ldots, m-1$.

*Subcase 1.* $h = 0$. From equation (7) with $i = 2m - 3$ we obtain

$$yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z} = -\sum_{i=0}^{m-1} 2v_1 a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^i + \sum_{i=0}^{m-1} 2i(v_3 - v_1)a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^{i-1}z^2.$$

From this equation and using the method of characteristic curves for solving linear partial differential equations, we obtain

$$\overline{f}_{2m-3} = \sum_{i=0}^{m-1} 2v_1 a_i^{m-1} u^{m-1-i} v^i \int \frac{\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} - \sum_{i=0}^{m-1} 2i(v_3 - v_1)a_i^{m-1}u^{m-1-i}v^{i-1}\int \frac{\pm\sqrt{v-w^2}}{\pm\sqrt{u-w^2}} + \overline{A}_{2m-3}(u, v).$$

In order that $f_{2m-3}$ is a homogeneous polynomial of degree $2m - 3$, we must have $\overline{A}_{2m-3}(x^2+y^2, y^2+z^2) \equiv 0$, $v_1 a_i^{m-1} = 0$ and $i(v_3 - v_1)a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-1$. Since $v_1 \neq v_3$, we obtain that $v_1 a_0^{m-1} = 0$ and $a_i^{m-1} = 0$ for $i = 1, 2, \ldots, m-1$.

If $v_1 = 0$, then $c = v_1 = v_2 = 0$, $v_3 \neq 0$, and

$$f_{2m-2} = a_0^{m-1}(x^2+y^2)^{m-1} \qquad f_{2m-3} \equiv 0.$$

By recursive calculations and in a similar way to solving $f_{2m-2}$ and $f_{2m-3}$ we obtain that for $l = 2, 3, \ldots, m-1$

$$f_{2m-2l} = a_0^{m-l}(x^2+y^2)^{m-l} \qquad f_{2m-2l-1} \equiv 0.$$

Therefore, we obtain the function

$$f = \sum_{i=1}^{m} a_0^i (x^2+y^2)^i$$

which is a polynomial first integral of degree $2m$, where $a_0^m \neq 0$ and $a_0^i$ is an arbitrary constant for $i = 1, 2, \ldots, m-1$. This proves statement (d) of theorem 1.

If $v_1 \neq 0$, then $a_0^{m-1} = 0$. So, the function $f_{2m-2} \equiv 0$. By recursive calculations, we can obtain that $f_{2m-l} \equiv 0$ for $l = 3, 4, \ldots, 2m$. Therefore, we obtain the function

$$f = a_0^m (x^2+y^2)^m$$

which is a Darboux polynomial with the constant cofactor $-2mv_1$. This proves statement (e) of theorem 1.

*Subcase 2.* $h \neq 0$, then $v_3 = 2v_1$. Since $v_2 = v_1$ and $v_3 \neq v_2$, this verifies that $v_1 \neq 0$. Introducing $f_{2m-2}$ into equation (7) with $i = 2m-3$ and performing some computations give

$$yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z} = -(4h)^3\frac{1}{3}\binom{m}{3} a_0^m (x^2+y^2)^{m-3}xyz^2 - \sum_{i=0}^{m-1} 2v_1 a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^i + \sum_{i=0}^{m-1} 2iv_1 a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^{i-1}z^2 - \sum_{i=0}^{m-2}[4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^{m-1}](x^2+y^2)^{m-2-i}(y^2+z^2)^i xy.$$

Using the changes (8) and (9), from this equation we obtain the following ordinary differential equation:

$$\frac{\mathrm{d}\overline{f}_{2m-3}}{\mathrm{d}w} = (4h)^3 \frac{1}{3}\binom{m}{3} a_0^m u^{m-3} w \left(\pm\sqrt{v-w^2}\right) + \sum_{i=0}^{m-1} 2v_1 a_i^{m-1} u^{m-1-i} v^i \frac{1}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} - \sum_{i=0}^{m-1} 2i v_1 a_i^{m-1} u^{m-1-i} v^{i-1} \frac{\pm\sqrt{v-w^2}}{\pm\sqrt{u-w^2}} + \sum_{i=0}^{m-2} [4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^{m-1}] u^{m-2-i} v^i \frac{w}{\pm\sqrt{v-w^2}}.$$

Integrating this equation with respect to $w$, we have

$$\overline{f}_{2m-3} = -(4h)^3 \binom{m}{3} a_0^m u^{m-3} \left(\pm\sqrt{v-w^2}\right)^3 + \sum_{i=0}^{m-1} 2v_1 a_i^{m-1} u^{m-1-i} v^i \int \frac{\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} - \sum_{i=0}^{m-1} 2i v_1 a_i^{m-1} u^{m-1-i} v^{i-1} \int \frac{\pm\sqrt{v-w^2}}{\pm\sqrt{u-w^2}}\,\mathrm{d}w - \sum_{i=0}^{m-2} [4h(m-1-i)a_i^{m-1} + 2h(i+1)a_{i+1}^{m-1}] u^{m-2-i} v^i \left(\pm\sqrt{v-w^2}\right) + \overline{A}_{2m-3}(u, v)$$

where $\overline{A}_{2m-3}$ is an arbitrary function in $u$ and $v$. In order that $f_{2m-3}$ is a homogeneous polynomial of degree $2m-3$, we must have $\overline{A}_{2m-3}(x^2+y^2, y^2+z^2) \equiv 0$, $v_1 a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-1$. This means that $a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-1$. Moreover, we have

$$f_{2m-3} = -(4h)^3 \binom{m}{3} a_0^m (x^2+y^2)^{m-3} z^3.$$

By recursive calculations we obtain that

$$f_{2m-s} = \begin{cases} (-1)^s (4h)^s \binom{m}{s} a_0^m (x^2+y^2)^{m-s} z^s & \text{for} \quad s = 4, 5, \ldots, m \\ 0, & \text{for} \quad s = m+1, m+2, \ldots, 2m. \end{cases}$$

Hence, the Darboux polynomial of degree $2m$ is

$$f = \sum_{s=0}^{m} (-1)^s (4h)^s \binom{m}{s} a_0^m (x^2+y^2)^{m-s} z^s = a_0^m (x^2+y^2-4hz)^m$$

with the cofactor $-2mv_1$. This proves statement (f) of theorem 1.

*Case (iii).* $v_1 \neq v_2$ and $v_2 = v_3$. Then $c = -2mv_2$, and

$$f_{2m} = a_m^m (y^2+z^2)^m \qquad f_{2m-1} = -2hm a_m^m (y^2+z^2)^{m-1} z.$$

Introducing $f_{2m-1}$ into equation (7) with $i = 2m - 2$ and performing some computations, we obtain

$$yz\frac{\partial f_{2m-2}}{\partial x} - xz\frac{\partial f_{2m-2}}{\partial y} + xy\frac{\partial f_{2m-2}}{\partial z} = 2hmv_2a_m^m(y^2+z^2)^{m-1}z$$
$$+(2h)^2 2\binom{m}{2}a_m^m(y^2+z^2)^{m-2}xyz.$$

From this equation and using the changes (8) and (9), we can obtain

$$\overline{f}_{2m-2} = -2hmv_2a_m^m v^{m-1}\arcsin\frac{w}{\sqrt{u}} - (2h)^2\binom{m}{2}a_m^m v^{m-2}w^2 + \overline{A}_{2m-2}(u,v).$$

In order that $f_{2m-2}$ is a polynomial, we must have $hv_2 = 0$.

*Subcase 1.* $h = 0$. Then

$$f_{2m-1} = 0 \qquad f_{2m-2} = \sum_{i=0}^{m-1} a_i^{m-1}(x^2+y^2)^{m-1-i}(y^2+z^2)^i.$$

Inserting $f_{2m-2}$ into equation (7) with $i = 2m - 3$ and performing some computations, we obtain

$$yz\frac{\partial f_{2m-3}}{\partial x} - xz\frac{\partial f_{2m-3}}{\partial y} + xy\frac{\partial f_{2m-3}}{\partial z}$$
$$= \sum_{i=0}^{m-1} a_i^{m-1}[2(m-1-i)v_1 + 2(i-m)v_2](x^2+y^2)^{m-1-i}(y^2+z^2)^i$$
$$+\sum_{i=0}^{m-1} a_i^{m-1}2(m-1-i)(v_2-v_1)(x^2+y^2)^{m-2-i}(y^2+z^2)^i y^2.$$

Working in a similar way to the proof of $f_{2m-1}$, we obtain from this equation that

$$\overline{f}_{2m-3} = -\sum_{i=0}^{m-1} a_i^{m-1}[2(m-1-i)v_1 + 2(i-m)v_2]u^{m-1-i}v^i$$
$$\times\int\frac{\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} - \sum_{i=0}^{m-1} a_i^{m-1}2(m-1-i)(v_2-v_1)$$
$$\times u^{m-2-i}v^i\int\frac{w^2\mathrm{d}w}{\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)} + \overline{A}_{2m-3}(u,v).$$

In order to obtain a homogeneous polynomial solution $f_{2m-3}$ of degree $2m - 3$, we must have $\overline{A}_{2m-3} \equiv 0$, and

$$[2(m-1-i)v_1 + 2(i-m)v_2]a_i^{m-1} = 0 \qquad (m-1-i)(v_2-v_1)a_i^{m-1} = 0$$

for $i = 0, 1, \ldots, m-1$. This means that $a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m-2$ and $v_2a_{m-1}^{m-1} = 0$. Moreover, we have $f_{2m-2} = a_{m-1}^{m-1}(y^2+z^2)^{m-1}$ and $f_{2m-3} \equiv 0$.

If $v_2 = 0$, then $c = v_2 = v_3 = 0$ and $v_1 \neq 0$. By recursive calculations and in a similar way to solve $f_{2m-2}$ and $f_{2m-3}$, we can obtain that for $l = 2, 3, \ldots, m-1$

$$f_{2m-2l} = a_{m-l}^{m-l}(y^2+z^2)^{m-l} \qquad f_{2m-2l-1} \equiv 0.$$

Therefore, we obtain the polynomial first integral of degree $2m$

$$f = \sum_{i=1}^{m} a^i(y^2+z^2)^i$$

where $a^m \neq 0$ and $a^i$ is an arbitrary constant for $i \neq 0$. This proves statement (g) with $h = 0$ of theorem 1.

If $v_2 \neq 0$, then $a_{m-1}^{m-1} = 0$. So, the function $f_{2m-2} \equiv 0$. By recursive calculations, we can prove that $f_{2m-l} \equiv 0$ for $l = 4, 5, \ldots, 2m$. Hence, we obtain the Darboux polynomial of degree $2m$:

$$f = a_m^m (y^2 + z^2)^m$$

with the cofactor $-2mv_2$. This proves statement (h) of theorem 1.

*Subcase 2.* $h \neq 0$ and $v_2 = 0$. then $c = v_3 = 0$ and $v_1 \neq 0$. Moreover, we have

$$f_{2m-2} = -(2h)^2 \binom{m}{2} a_m^m (y^2 + z^2)^{m-2} y^2 + \overline{A}(x^2 + y^2, y^2 + z^2)$$

$$= (2h)^2 \binom{m}{2} a_m^m (y^2 + z^2)^{m-2} z^2 + \sum_{i=0}^{m-1} a_i^{m-1} (x^2 + y^2)^{m-1-i} (y^2 + z^2)^i.$$

Introducing $f_{2m-2}$ into equation (7) with $i = 2m - 3$ and performing some computations give

$$yz \frac{\partial f_{2m-3}}{\partial x} - xz \frac{\partial f_{2m-3}}{\partial y} + xy \frac{\partial f_{2m-3}}{\partial z} = -(2h)^3 3 \binom{m}{3} a_m^m (y^2 + z^2)^{m-3} xyz^2$$

$$+ \sum_{i=0}^{m-1} a_i^{m-1} 2(m - 1 - i) v_1 (x^2 + y^2)^{m-2-i} (y^2 + z^2)^i x^2$$

$$- \sum_{i=0}^{m-2} [4h(m - 1 - i) a_i^{m-1} + 2h(i + 1) a_{i+1}^{m-1}] (x^2 + y^2)^{m-2-i} (y^2 + z^2)^i xy.$$

From this equation and using the changes (8) and (9), we can obtain the following solution:

$$\overline{f}_{2m-3} = -(2h)^3 \binom{m}{3} a_m^m v^{m-3} \left(\pm\sqrt{v - w^2}\right)^3$$

$$- \sum_{i=0}^{m-1} a_i^{m-1} 2(m - 1 - i) v_1 u^{m-2-i} v^i \int \frac{\pm\sqrt{u - w^2}}{\pm\sqrt{v - w^2}} \, \mathrm{d}w$$

$$- \sum_{i=0}^{m-2} [4h(m - 1 - i) a_i^{m-1} + 2h(i + 1) a_{i+1}^{m-1}] u^{m-2-i} v^i \left(\pm\sqrt{v - w^2}\right)$$

$$+ \overline{A}_{2m-3}(u, v)$$

where $\overline{A}_{2m-3}$ is an arbitrary function in $u$ and $v$. In order that $f_{2m-3}(x, y, z) = \overline{f}_{2m-3}(u, v, z)$ is a homogeneous polynomial of degree $2m - 3$, we must have $\overline{A}_{2m-3}(x^2 + y^2, y^2 + z^2) \equiv 0$ and

$$(m - 1 - i) a_i^{m-1} = 0 \qquad \text{for} \quad i = 0, 1, \ldots, m - 1.$$

This means that $a_i^{m-1} = 0$ for $i = 0, 1, \ldots, m - 2$. Therefore, we obtain that

$$f_{2m-2} = (2h)^2 \binom{m}{2} a_m^m (y^2 + z^2)^{m-2} z^2 + a_{m-1}^{m-1} (y^2 + z^2)^{m-1}$$

$$f_{2m-3} = -(2h)^3 \binom{m}{3} a_m^m (y^2 + z^2)^{m-3} z^3 - 2h(m - 1) a_{m-1}^{m-1} (y^2 + z^2)^{m-2} z.$$

Working in a similar way to subcase 2 of case (i), we can obtain recursively that the Darboux polynomial of degree $2m$ is

$$f = \sum_{i=1}^{m} a_i (y^2 + z^2 - 2hz)^i$$

where $a_m \neq 0$ and $a_i$ are arbitrary constants for $i \neq 0$. This proves statement (g) of theorem 1.

*Case (iv).* $v_1 \neq v_2$ and $v_2 \neq v_3$. Then $v_1 = v_3$, $c = -2mv_1$ and $(m-i)a_i^m + (i+1)a_{i+1}^m = 0$ for $i = 0, 1, \ldots, m-1$, that is

$$a_i^m = (-1)^i \binom{m}{i} a_0^m \qquad i = 1, 2, \ldots, m.$$

Therefore, we have

$$f_{2m} = \sum_{i=0}^{m} a_i^m (x^2+y^2)^{m-i}(y^2+z^2)^i = a_0^m (x^2-z^2)^m$$

$$f_{2m-1} = -2hma_0^m (x^2-z^2)^{m-1} z.$$

Introducing $f_{2m-1}$ into equation (7) with $i = 2m-2$ and working in a similar way to solve $f_{2m-1}$, we can prove that

$$f_{2m-2} = (2h)^2 \binom{m}{2} a_0^m (x^2-z^2)^{m-2} z^2 + \sum_{i=0}^{m-1} a_i^{m-1} (x^2+y^2)^{m-1-i}(y^2+z^2)^i$$

with the condition $hv_1 = 0$.

*Subcase 1.* $h = 0$. Then working in a similar way to the proof of subcase 1 of case (iii), we can obtain that if $v_1 = 0$ the Darboux polynomial of degree $2m$ is

$$f = \sum_{i=1}^{m} a_i (x^2-z^2)^i$$

where $a_m \neq 0$ and $a_i$ is an arbitrary constant for $i \neq m$. This proves statement (j) with $h = 0$ of theorem 1.

If $v_1 \neq 0$, the Darboux polynomial of degree $2m$ is

$$f = a_0^m (x^2-z^2)^m$$

with the cofactor $-2mv_1$. This proves statement (i) of theorem 1.

*Subcase 2.* $h \neq 0$. Then $v_1 = v_3 = c = 0$ and $v_2 \neq 0$. Then working in a similar way to the proof of subcase 2 of case (i) and of subcase 2 of case (iii), we can prove that the Darboux polynomial of degree $2m$ is

$$f = \sum_{i=1}^{m} a_i (x^2-z^2-2hz)^i$$

where $a_m \neq 0$ and $a_i$ is an arbitrary constant for $i \neq m$. This proves statement (j).

Combining all the results we complete the proof of the theorem.

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